Integral treatment of buoyancy-induced flows in a porous medium adjacent to horizontal surfaces with variable wall temperature

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An integral treatment based on the Kärman-Pohlhausen integral relation is proposed for the analysis of the buoyancy-induced flows in a porous medium adjacent to horizontal surfaces with variable wall temperature. It is shown that a significant improvement of its accuracy is possible by matching a compatibility condition on the second derivative of the temperature profile at the wall. A comparison of the present approximate solution and the exact solution reveals an excellent performance of the present approximate solution procedure.

Keywords: *porous media; boundary layer; integral method; buoyancy*

Introduction

A great deal of attention has been directed toward the study of buoyancy-induced flows within a porous medium in view of its applications in both engineering and geothermal systems. For example, hot dike complexes in a volcanic region can provide the energy source for the heating of ground water that can be used for power generation. Thus the study of heat transfer characteristics of buoyancy-induced flows in a porous medium is essential to assessing and evaluating geothermal resources.

When the wall temperature of a semi-infinite horizontal upward facing plate is kept higher than that of a surrounding porous medium, a vertical density gradient is generated within the thermal boundary layer over the flat plate, which then will create a longitudinal pressure gradient. If the pressure force is greater than the buoyancy force, the fluid moves along the horizontal plate so as to relax its pressure. The problem has many important applications such as to convective flows above the heated bedrock and below the cooled caprock in a liquid-dominant geothermal reservoir.

Recently, Cheng[1] attacked the problem applying the Kärman-Pohlhausen integral relation and obtained similarity solutions assuming the exponential variation of the wall temperature. Neither the agreement of these approximate solutions nor the exact solutions obtained by Cheng and Chang,[2] however, seems satisfactory. It is the purpose of this paper to show that a significant improvement of its accuracy can be achieved by matching a compatibility condition on the second derivative of the temperature profile at the wall, which is implicit in the energy conservation equation in a differential form. The convective heat transfer coefficient obtained in this study will be quite useful in estimating the cooling rate of horizontal bedrocks trapped in an aquifer as well as the heat loss rate from underground energy transport and storage systems.

Analysis

The physical model and coordinates are indicated in Figure 1. A semi-infinite horizontal surface is heated to the temperature $T_w(x)$ above the ambient temperature T_e of the surrounding porous medium. The governing equations in the boundary layer coordinates (x, y) are given by

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \qquad (1)$$

$$\frac{\partial u}{\partial y}=-\frac{Kg\beta}{\nu}\frac{\partial T}{\partial x} \qquad (2)$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\kappa\frac{\partial^2 T}{\partial y^2} \qquad (3)$$

where:
K the permeability of the porous medium
g the acceleration due to gravity
β the coefficient of thermal expansion
ν the kinematic viscosity
κ the equivalent thermal diffusivity of the porous medium.

Equations 1, 2, 3 represent the continuity equation, the Darcy's law with the Boussinesq approximation, and the energy equation, respectively. The boundary conditions are

$$y=0: \quad v=0 \quad T=T_w(x) \qquad (4a, b)$$

$$y\rightarrow\infty: \quad u=0 \quad T=T_e \qquad (4c, d)$$

Equation 2 may be integrated across the boundary layer of thickness δ, using the boundary conditions given by Equations (4a–d), so that

$$u_w=\frac{Kg\beta}{\nu}\frac{d}{dx}\int_0^\delta (T-T_e)\,dy \qquad (5)$$

Similarly integrating the energy equation (Equation 3) with the

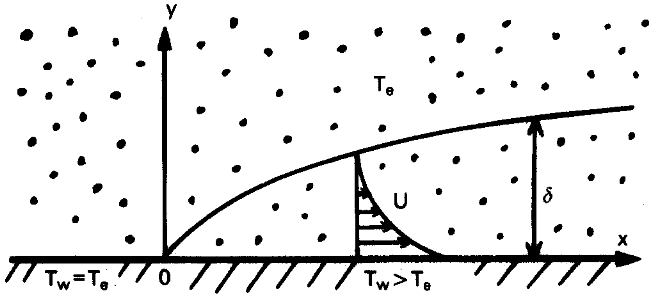

Figure 1 Physical model and coordinates

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continuity equation (Equation 1) leads to

$$\frac{d}{dx}\int_0^\delta u(T-T_e)\,dy=-\kappa\left.\frac{\partial T}{\partial y}\right|_{y=0} \tag{6}$$

Now, considering Equation 3 at the impermeable wall gives the following auxiliary relation:

$$u_w\frac{dT_w}{dx}=\kappa\left.\frac{\partial^2 T}{\partial y^2}\right|_{y=0} \tag{7}$$

This compatibility condition associated with the temperature profile curvature at the wall is not trivial, since the local temperature gradient at the wall determines the heat transfer rate there. Introducing this auxiliary relation leads to a significant improvement of the accuracy of the integral method.

To seek similarity solutions, a power-law variation is prescribed for the wall temperature, namely,

$$\Delta T_w\equiv T_w-T_e\propto x^n \tag{8}$$

On noting the foregoing relation, Equations 5, 6, and 7 may be reduced to the following algebraic equations:

$$u_w=\frac{Kg\beta}{\nu}A(n+m)\frac{\Delta T_w\delta}{x} \tag{9}$$

$$(2n+2m-1)B\frac{u_w\delta}{x}=C\frac{\kappa}{\delta} \tag{10}$$

$$n\frac{u_w}{x}=D\frac{\kappa}{\delta^2} \tag{11}$$

where

$$A=\int_0^1 f\,d\eta \qquad B=\int_0^1\left(\frac{u}{u_w}\right)f\,d\eta \qquad C=-\left.\frac{\partial f}{\partial\eta}\right|_{\eta=0} \tag{12a, b, c}$$

$$D=\left.\frac{\partial^2 f}{\partial\eta^2}\right|_{\eta=0} \qquad f(\eta)\equiv\frac{T-T_e}{\Delta T_w} \qquad \eta\equiv\frac{y}{\delta} \tag{12d, e, f}$$

The shape factors A through D can be determined as the profiles for u/u_w and f are specified. Moreover, m is the unknown exponent describing the growth of the boundary layer so that

$$\delta\propto x^m \tag{13}$$

Substituting Equation 9 into Equations 10 and 11 yields the following two distinct expressions for δ^3:

$$(\delta/x)^3\,\text{Rax}=\frac{C}{(2n+2m-1)(n+m)AB}=\frac{D}{n(n+m)A} \tag{14a}$$

where

$$\text{Rax}=\frac{Kg\beta\,\Delta T_w x}{\kappa\nu} \tag{14b}$$

The constancy of the right-hand side of Equation 14a indicates that

$$m=\frac{2-n}{3} \tag{15}$$

The next step is to specify the functional forms for the velocity and temperature profiles. The velocity profile may be approximated by a second-order polynomial as

$$\frac{u}{u_w}=(1-\eta)^2 \tag{16}$$

and the temperature profile may be given by the following simple one-parameter profile family:

$$f(\eta;\alpha)=(1-\eta)^\alpha \tag{17}$$

The shape factors A through D may readily be evaluated by substituting Equations 16 and 17 into Equations 12a through 12d:

$$A=\frac{1}{1+\alpha} \qquad B=\frac{1}{3+\alpha} \qquad C=\alpha \qquad D=\alpha(\alpha-1) \tag{18a, b, c, d}$$

Substituting the foregoing expressions along with Equation 15 into the two expressions on the right-hand side of Equation 14a yields the following explicit relation between the profile parameter α and the power-law exponent n for the wall temperature variation:

$$\alpha=\frac{1+13n}{1+n} \tag{19}$$

Equation 14a with α given by the foregoing expression finally

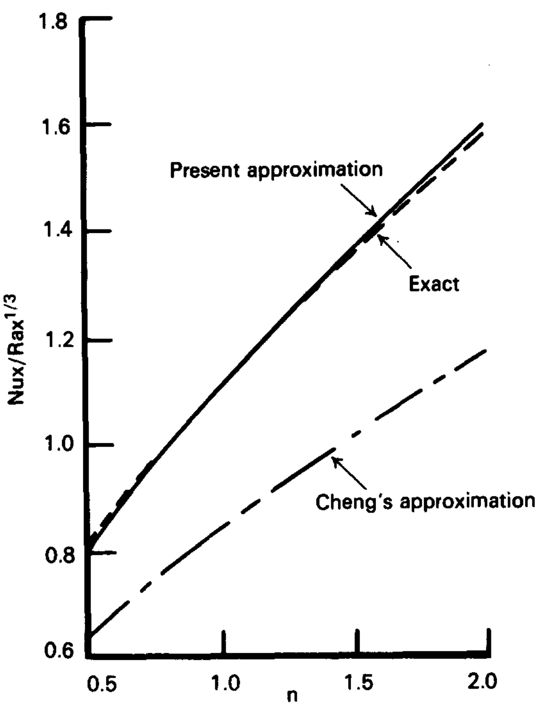

Figure 2 Local heat transfer results

Notation

A, B, C, D	Shape factors	T	Temperature
f	Profile function	u, v	Velocity components in x and y directions
g	Acceleration due to gravity	x, y	Boundary layer coordinates
K	Permeability	α	Shape parameter
m	Parameter associated with boundary layer growth	β	Coefficient of thermal expansion
		δ	Boundary layer thickness
n	Power-law exponent of the wall temperature	η	Similarity variable
Nux	Local Nusselt number	κ	Thermal diffusivity of the porous medium
Rax	Local Rayleigh number	ν	Kinematic viscosity

Table 1 $Nux/Rax^{1/3}$

n	Present approximation	Exact number	Cheng's approximation
0.5	0.8046	0.8164	0.6436
1.0	1.108	1.099	0.8399
1.5	1.364	1.351	1.012
2.0	1.594	1.571	1.169

gives the expression for the local Nusselt number $Nux = (-x/\Delta T_w)(\partial T/\partial y)|_{y=0} = \alpha(x/\delta)$ of the present interest as

$$Nux/Rax^{1/3} = \left[\frac{(1+n)(1+13n)^2}{36(1+7n)}\right]^{1/3} \tag{20}$$

Results

Thus the local heat transfer rate is readily calculable from Equation 20 for any given exponent n for the wall temperature variation. It should, however, be pointed out that the range of n for which the problem is physically realistic is restricted to $1/2 \leq n \leq 2$, since both u_w and δ must increase with respect to x (see Equations 9 and 15).

The values based on Equation 20 are plotted in Figure 2 and are also tabulated in Table 1 for a direct comparison with those from the exact solution[2] and the integral method by Cheng.[1] Agreement between the present approximate solution and the exact solution appears to be excellent. The significant improvement in its accuracy achieved in this approximate procedure is due to the effort made to satisfy the compatibility condition associated with the second derivative of the temperature profile at the wall, namely, Equation 7.

References

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